

**ADAPTIVE METHODS, ROLLING CONTACT,  
AND NONCLASSICAL FRICTION LAWS**

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**ABSTRACT**

Results and methods on three different areas of contemporary research are outlined. These include adaptive methods, the rolling contact problem for finite deformation of a hyperelastic or viscoelastic cylinder, and non-classical friction laws for modeling dynamic friction phenomena.

## 1. INTRODUCTION

This paper addresses three subjects that impact on the computer simulation of nonlinear tire behavior: adaptive methods, which represent schemes for assessing numerical error and automatically adapting the mesh so as to improve accuracy; the rolling contact problem, which is at the heart of tire analysis; and new friction laws, which are essential in developing realistic models of frictional contact. Space limitations preclude a detailed discussion of these issues; but further details can be found in recent papers and reports by the author and his colleagues [1-17].

## 2. ROLLING CONTACT

The general rolling contact problem as a basis for nonlinear tire analysis involves some of the most challenging and difficult problems in structural mechanics. Among the complicating features are the presence of unilateral contact, friction, inertia effects, multi-parameter bifurcations, the emergence of standing waves, viscoelastic and thermal effects, large deformations, the necessity of modeling of complex materials such as fiber-reinforced rubbers, and the presence of non-conservative pressurization loadings. A first step toward resolving such issues is the formulation of correct kinematics and variational principles for a special case: the steady-state rolling of a hyperelastic or viscoelastic cylinder in contact with a rigid foundation and in a state of plane strain.

The kinematical situation is shown in Fig. 1 where the geometry of the reference configuration (a rigid spinning cylinder with no contact) is compared to the geometry of a deformed cylinder in steady-state rolling contact with a rough (frictional) roadbed (foundation).

Key features of the kinematic description are listed as follows:

1) Time appears only implicitly in the formulation; if  $(R, \Theta, Z)$  are material coordinates, the referential coordinates are

$$r = R, \quad \theta = \Theta + \omega t, \quad z = Z$$

where  $\omega$  is the angular velocity of the rigid, reference cylinder.

2) If the motion is defined by the map

$$x_i = \chi_i(r, \theta, z) \quad i = 1, 2, 3$$

where  $x_i$  = spatial Cartesian coordinates of particles in the current configuration, then velocity and acceleration components are

$$v_i = \omega \frac{\partial \chi_i}{\partial \theta} \quad a_i = \omega^2 \frac{\partial^2 \chi_i}{\partial \theta^2}$$

3) The unilateral contact constraint can be expressed in the form

$$\chi_2 \leq H \text{ on } \Gamma_C$$

where  $\Gamma_C$  is the exterior contact surface and  $H$  is the distance from the hub center to the foundation. This condition can also be written

$$(\chi_2 - H)_+ = 0$$

where  $(\cdot)_+$  denotes the positive part of  $(\cdot)$ .

4) The time history of deformation can be expressed in terms of strains of particles located on the same circular arc in the reference configuration. For example, if  $\tilde{E}$  is the Green-Saint Venant strain tensor, its time history over an interval  $0 \leq \tau \leq t$  satisfies:

$$\{\tilde{E}(r, \theta, z, \tau); 0 \leq \tau \leq t\} = \{\tilde{E}(r, \chi, z, t), 0 \leq \chi \leq \omega t\}$$

This property makes it possible to incorporate viscoelastic effects into the rolling contact problem in a straightforward way.

For illustration purposes, we consider a class of rolling contact problems in which the following constitutive properties hold:

a) The material is either an isotropic hyperelastic material characterized by a strain energy function

$$W = W(I_1, I_2, I_3)$$

(or  $W = W(I_1, I_2)$  if  $I_3 = 1$  - an incompressible material) in which  $I_1, I_2, I_3$  are the principal invariants of the deformation tensor  $\tilde{C} = \tilde{F}^T \tilde{F}$ , or the material is a viscoelastic material characterized by a linear viscoelastic perturbation of a hyperelastic material; e.g.,

$$\tilde{S} = \frac{\partial W}{\partial \tilde{E}} + \gamma \int_{-\infty}^t k(\tau, t) \dot{\tilde{E}}(\tau) d\tau$$

with  $\gamma$  a viscosity parameter,  $k$  a material kernel, and  $\tilde{S}$  the second Piola-Kirchhoff stress tensor.

b) The normal stiffness of the contact interface obeys a constitutive law of the type

$$t_n = c_n (\chi_2 - H)^{m_n} \text{ on } \Gamma_C$$

where  $t_n$  is the normal stress and  $c_n$  and  $m_n$  are material constants. If  $m_n = 1$  and  $c_n = 1/\epsilon$ , where  $\epsilon$  is a positive constant, this relation coincides with the normal contact stress associated with an exterior penalty approximation of the unilateral constraint condition.

c) If the cylinder is given a prescribed velocity  $v_0$  relative to the roadway, the slip velocity on the contact surface is

$$w_T = v_0 - \dot{\chi}_1 = v_0 - w \partial_\theta \chi$$

5) The friction law is (with  $t_T$  the frictional stress)

$$|t_T| < \mu |t_n| \Rightarrow w_T = 0$$

$$|t_T| = \mu |t_n| \Rightarrow w_T = -\lambda |t_T| \text{ for some } \lambda \geq 0$$

Variational principles for various rolling contact problems are summarized in Figs. 2-8, beginning with the pure spinning of a cylinder without contact and progressing to the general variational inequality for finite deformation rolling contact with friction. Various spaces of admissible functions are defined in these figures as well as several nonlinear forms. In particular,

$A(\chi, \eta)$	= the internal virtual work produced by the Piola-Kirchhoff stresses $T_0$
$B(\chi, \eta)$	= the virtual work produced by inertia (radial acceleration) effects per unit of angular velocity $\omega$
$C(\chi, \eta)$	= work term due to normal compliance of the interface
$I(\chi, \xi, \eta)$	= a virtual work term representing the work done by the hydrostatic pressure $p$ (present when the material is incompressible)
$j(\chi, \eta)$	= the virtual work term due to frictional forces
$f(\eta)$	= the virtual work due to external forces

A finite-element code has been developed based on this general variational principle which has the following features:

1. Biquadratic ( $Q_2$ ) elements are used to approximate the displacement field and, for incompressible materials,  $P_1$ , discontinuous linear elements are used to approximate hydrostatic pressures
2. The frictional functions are regularized in a standard way
3. A Riks-Crisfield method with Newton-Raphson corrections is used to solve the nonlinear systems of equations characterizing the discrete problem

To date, an extensive set of numerical solutions has been obtained using these concepts and methods. Here only one example is cited, which is interesting because of the slow emergence of standing waves as the angular velocity is increased for a fixed penetration  $H$  of a hollow rubber cylinder into a rough roadway with coefficient of friction  $\mu = 0.03$ . Computed deformed shapes and stress contours are shown in Figs. 9-13 for various values of  $\omega$ . We notice the steady development of more-or-less periodic wavelets on the exterior surface which meet at points at which singularities appear to be forming. The presence of friction on the contact surface destroys the symmetry of this wavelet pattern. Mild viscoelastic effects, such as those in rubbers at moderate temperatures, do not appreciably alter the structure of these deformed geometries.

The generality of the formulation and of the methods employed here makes it possible to study numerically a wide range of rolling contact problems. Further work shall involve generalizations of these results to three-dimensional rolling contact problems which include the effects of turning, steering forces, and tilting relative to the roadway plane.

### 3. ADAPTIVE METHODS

We shall now turn to the important subject of adaptive finite-element methods. Adaptive methods should have a significant impact on not only tire analysis but also on general computational structural mechanics in the relatively near future.

In general, adaptive methods seek to change the structure of an approximate method to improve the quality of the solution. By structure, we mean the mesh density, locations of nodes, and order of the local polynomials. By quality of an approximation, we generally mean the error in approximation in some appropriate norm. There are, thus, two primary aspects of any adaptive finite-element method:

- 1) The estimation of the error
- 2) The adaptive strategy

In the first of these, it is assumed that an initial approximation of the solution is known, perhaps from a computation on a coarse mesh, and that this rough solution can be used to obtain an a posteriori estimate of the local

error over each finite element. Once an estimate of the local error is known, one must call upon some technique to reduce the local error and thereby improve the quality of the solution.

There are two general types of methods we have studied for a posteriori error estimation of the local error over each finite element. Once an estimate of the local error is known, one must call upon some technique to reduce the local error and thereby improve the quality of the solution.

There are two general types of methods we have studied for a posteriori error estimation:

1. Residual methods
2. Interpolation (or a priori) methods

As the name implies, residual methods make use of element residuals - the residual or unbalance left over when the approximate solution is substituted into the governing equations and edge conditions on each element.

The residual itself (e.g., the equilibrium unbalance in element forces) is not necessarily a good indication of local error. Indeed, the local residual can be nearly zero while the error can be quite large. For this reason, it is generally necessary to calculate certain local error indicators  $\phi_K$  which bound the error above and below in appropriate norms. The calculation of error indicators generally requires the solution of special local problems over each element in which the element residuals enter as data. For example, in the model elliptic problem,

$$-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2$$

$$u = 0 \text{ on } \partial\Omega$$

(with  $\Delta$  the Laplacian and  $f$  given), the finite-element solution  $u_h$  satisfies

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx dy = \int_{\Omega} f v_h \, dx dy$$

for arbitrary test function  $v_h$ , and over each element  $K$  of a mesh, the residual is

$$r_h = -\Delta u_h - f$$

Over element  $K$ , an error indicator  $\phi_K$  is computed which satisfies

$$\int_K \nabla \phi_K \cdot \nabla v \, dx dy = \int_K r_h v \, dx dy + \oint_K \phi_K \frac{\partial u_h}{\partial n} v \, ds$$

for  $v$  in  $H^1(K)$ . One can show that the error  $e_h = u - u_h$  in the energy norm ( $\|e_h\|_{1,\Omega} = \{\int_{\Omega} |\nabla e_h|^2 \, dx dy\}^{1/2}$ ) satisfies the bound

$$\|e_h\|_{1,\Omega}^2 \leq \sum_K \|\phi_K\|_{1,K}^2$$

Various residual methods differ in the way these error indicators are defined and calculated. There are some residual techniques which can produce sharp local error estimates in virtually any norm for certain classes of problems. (See Demkowicz and Oden [4, 5]). These methods are not restricted to linear problems and have been used to produce error estimates in highly non-linear problems (see [7, 16]).

The interpolation methods make use of the fact that the interpolation error over an element  $K$  of diameter  $h_K$  over an element on which polynomials of degree  $p$  are used is

$$|u - \Pi_h u|_{1,K}^2 \leq C h_K^{2p} |u|_{2,K}^2$$

where  $u$  is a given smooth function,  $\Pi_h u$  is its interpolant,

$$|u|_{1,K}^2 = \int_K |\nabla u|^2 \, dx dy$$

$$|u|_{2,K}^2 = \int_K (u_{xx}^2 + u_{xy}^2 + u_{yy}^2) \, dx dy$$

and  $C$  is a constant independent of  $h_K$  and  $u$ . The idea behind interpolation methods is to estimate  $|u|_{2,K}^2$  using results of a coarse-mesh approximation (e.g., using finite-difference methods or extraction methods [6]). The constant  $C$  cannot, in general, be determined, so such interpolation methods can only be used to assess relative error in various finite elements.

Once an estimate of the error is obtained, the local error is reduced adaptively using one of the following techniques:

1. h-methods: the mesh size  $h$  is reduced and the number of elements is increased in regions of high error.

2. p-methods: the mesh is fixed, but the local order  $p$  of the polynomial shape functions is increased.

3. Moving mesh methods: the nodes in a finite-element mesh are moved and concentrated in areas of high error.

4. Combined methods: these involve combinations of the above three techniques.

We have developed test codes which employ all of these methods. The results of some tests are given in Figs. 14-20, and specific comments follow.

1. In Fig. 14 we see a distorted mesh obtained using a moving mesh strategy on the driven cavity problem for an incompressible viscous fluid (see [7]).

2. Figures 15, 16, and 17 contain computed results in adaptive schemes based on residual methods devised by Demkowicz and Oden [4, 5]. The results shown here are for transient heat conduction problems with dominate convection effects and for nonlinear Burgers' equation vector-valued solutions which simulate the Navier-Stokes equations. A special h-method is used here which employs a fine grid and a coarse-grid approximation to estimate error.

3. Results of a new p-method for Navier-Stokes equations in two-dimensions are illustrated in Figs. 18, 19 (see [1, 16]). Here a rather coarse mesh is used and errors are reduced by increasing local polynomial degrees from 1 to 2 to 3. Different shading in these figures indicates different levels of local  $L^2$  error, with black cells indicating an error of less than 5 percent, grey an error of less than 10 percent, and white an error of over 20 percent. Such large local errors are reduced before the solution is allowed to advance in time. Remarkably, the so-called effectivity index  $\theta$  for this problem (which represents the ratio of the estimated local error to the actual local error) for an  $L^2$  - norm varied from around 1.001 to 0.860 for the time ranges considered in a test example. This suggests that residual-type error estimates based on p-type strategies can be very accurate, even for transient nonlinear problems on coarse meshes.

Figure 20 shows refined mesh patterns for a class of linear elliptic problems in which a very fast vectorizable h-method is used in conjunction with an interpolation-type error estimator (see [6]). One interesting aspect of this work, indicated by different shadings of elements in the figure, is that the distinction between "optimal" meshes determined using a very sophisticated error estimator (see [17]) and very crude estimates ([6]) is negligible whenever strong singularities are present.

#### 4. NON-CLASSICAL FRICTION LAWS

In our most recent calculations of rolling contact, we have employed special interface constitutive equations for the normal compliance of the



interface and the tangential frictional forces. Some of these laws are mentioned in Section 2 of this paper (see also Fig. 6). The physical interpretation and the motivation of such models are discussed in references [14, 15, and 18].

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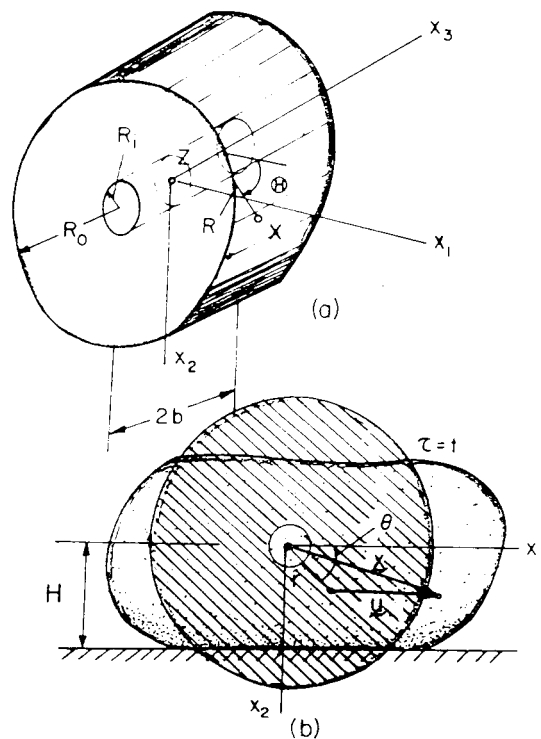


Figure 1

# STEADY SPINNING OF A DEFORMABLE CYLINDER

$\mathcal{V}$  = SPACE OF ADMISSIBLE MOTIONS

$$\equiv \{ \eta \in V \mid \eta|_{(r,\theta)} = R_i \text{ a.e. on } \Gamma_D \}$$

$$V \equiv \{ \eta = (\eta_1, \eta_2) \mid \left| \int_{\Omega_0} W(\nabla \eta) dv_0 \right| < \infty \}$$

CASE:  $W$  = POLYNOMIAL OF  $I_C, II_C, III_C$

$V$  = SOBOLEV SPACE OF ORDER  $(1,p)$

$$= (W^{1,p}(\Omega_0))^2 \quad \text{for some } p, \quad 2 \leq p < \infty$$

CASE: FOR VISCOELASTIC MATERIAL

$$U \equiv V \cup \{ \eta \in V \mid$$

$$\left| \int_{\Omega_0} F_{\tau=-\infty}^t (\tau, E(\nabla \eta)) : \nabla \eta dv_0 \right| < \infty \}$$

Figure 2

### VARIATIONAL PROBLEM $\equiv$ B.V.P

#### VARIATIONAL PROBLEM

FIND A MOTION  $\chi \in \psi$  SUCH THAT

$$A(\chi, \eta) = \omega^2 B(\chi, \eta) + f(\eta) \quad \text{for all } \eta \in \psi$$

#### BOUNDARY VALUE PROBLEM

$$\text{Div} T_0(\chi) + \rho_0 b_0 = \rho_0 \partial_{\theta}^2 \chi \quad \text{in } \Omega_0$$

$$\chi = R_i \quad \text{on } \Gamma_D$$

$$T_0(\chi) n_0 = 0 \quad \text{on } \Gamma/\Gamma_D = \Gamma_C$$

$$A(\chi, \eta) = \int_{\Omega_0} (\partial W(\nabla \chi) / \partial \nabla \chi) : \nabla \eta \, dv_0$$

$$\text{or } = \int_{\Omega_0} T_0(\chi) : \nabla \eta \, dv_0$$

$$T_0(\chi) = \nabla \chi S(\chi) = \text{1ST PIOLA-KIRCHHOFF STRESS}$$

$$B(\chi, \eta) = \int_{\Omega_0} \partial_{\theta} \chi \cdot \partial_{\theta} \eta \, dv_0$$

$$\partial_{\theta} \chi \cdot \partial_{\theta} \eta = (\partial \chi_i / \partial \theta \chi \partial \eta_i / \partial \theta)$$

$$f(\eta) = \int_{\Omega_0} \rho_0 b_0 \cdot \eta \, dv_0$$

Figure 3

### ROLLING CONTACT WITHOUT FRICTION

$\mathfrak{X} = \text{UNILATERAL CONSTRAINT SET}$

$$\equiv \{ \eta \in \psi \mid \eta_2 \leq H \text{ on } \Gamma_C \}$$

#### VARIATIONAL INEQUALITY

FIND A MOTION  $\chi \in \mathfrak{X}$  SUCH THAT

$$A(\chi, \eta - \chi) - \omega^2 B(\chi, \eta - \chi) \geq f(\eta - \chi) \quad \text{for all } \eta \in \mathfrak{X}$$

#### INCOMPRESSIBLE MATERIAL

$$\mathfrak{X} = \{ \eta \in \psi \mid \eta_2 \leq H \text{ on } \Gamma_C, \det \nabla \eta = 1 \text{ a.e. in } \Omega_0 \}$$

Figure 4

## ROLLING CONTACT WITH FRICTION

$\mathcal{U}$  = SPACE OF VELOCITY-MOTIONS

$$\equiv \{ \eta \in \mathcal{V} \mid (\omega \partial_\theta \eta_1, \eta_2) \in \mathcal{V} \}$$

VIRTUAL POWER BY FRICTION

$$j : \mathcal{V} \times \mathcal{U} \rightarrow \mathcal{R}$$

$$j(\chi, \eta) \equiv \int_{\Gamma_C} \mu |T_\theta(\chi) n_0 \cdot n| |\omega \partial_\theta \eta_1 - v_0| ds_0$$

VARIATIONAL INEQUALITY  $\equiv$  B.V.P.

VARIATIONAL INEQUALITY

FIND A MOTION  $\chi \in \mathcal{U} \cap \mathcal{K}$  SUCH THAT

$$A(\chi, \eta - \nabla_\theta^{-1} \chi) - \omega^2 B(\chi, \eta - \nabla_\theta^{-1} \chi) + j(\chi, \eta) -$$

$$j(\chi, \nabla_\theta^{-1} \chi) \geq f(\eta - \nabla_\theta^{-1} \chi) \quad \text{for all } \eta \in \mathcal{K}$$

$$\text{with } \nabla_\theta^{-1} \chi \equiv (\omega \partial_\theta \chi_1, \chi_2)$$

Figure 5

## GENERALIZATION TO NONCLASSICAL NORMAL CONTACT/FRICTION LAWS AND REGULARIZED FRICTION FUNCTIONALS

NORMAL COMPLIANCE ON CONTACT SURFACE

$$C : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$$

$$C(\chi, \eta) \equiv \int_{\Gamma_C} f(\chi_2, \omega \partial_\theta \chi_2) \eta_2 ds_0$$

$$f(\chi_2, \omega \partial_\theta \chi_2) = c_n (\chi_2 - H)_+^{m_n} +$$

$$b_n (\chi_2 - H)_+^{l_n} \dot{\chi}_2$$

$$\dot{\chi}_2 = \omega(\partial \chi_2 / \partial \theta)$$

VARIATIONAL INEQUALITY WITH NORMAL COMPLIANCE

FIND A MOTION  $\chi \in \mathcal{U}$  SUCH THAT

$$A(\chi, \eta - \nabla_\theta^{-1} \chi) - \omega^2 B(\chi, \eta - \nabla_\theta^{-1} \chi) + C(\chi, \eta - \nabla_\theta^{-1} \chi)$$

$$+ j(\chi, \eta) - j(\chi, \nabla_\theta^{-1} \chi) \geq f(\eta - \nabla_\theta^{-1} \chi)$$

Figure 6

### LAGRANGE MULTIPLIER METHOD

$\mathcal{Q}$  = SPACE OF MULTIPLIER

$$\det \nabla \eta \in L^p(\Omega_0) \longrightarrow \mathcal{Q} = L^{p'}(\Omega_0)$$

$$(1/p) + (1/p') = 1$$

=====

FIND  $(\chi, p) \in \mathcal{X} \times \mathcal{Q}$  SUCH THAT

$$A(\chi, \eta - \chi) - \omega^2 B(\chi, \eta - \chi) + I(p, \chi, \eta - \chi) \geq f(\eta - \chi)$$

for all  $\eta \in \mathcal{X}$

$$(q, (\det \nabla \chi - 1)) = 0$$

for all  $q \in \mathcal{Q}$

=====

$I(p, \chi, \eta)$  = TRILINEAR FORM

$$\equiv \int_{\Omega_0} p \operatorname{adj} \nabla \chi : \nabla \eta \, dv_0$$

$$(q, (\det \nabla \chi - 1)) \equiv \int_{\Omega_0} q (\det \nabla \chi - 1) \, dv_0$$

Figure 7

### REGULARIZED FRICTION LAW

$$J_\epsilon : \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{R}$$

$$J_\epsilon(\chi, \eta) \equiv \int_{\Gamma_C} \mu |T_0(\chi) n_0 \cdot \eta|$$

$$\varphi_\epsilon(|\omega \partial_\theta \chi_1 - v_0|) \eta_1 ds_0$$

$$\text{with } \varphi_\epsilon(|w_t|) = 1 \quad \text{if } |w_t| > \epsilon$$

$$= |w_t|/\epsilon \quad \text{if } |w_t| \leq \epsilon$$

$$\text{or} \quad = \tanh(|w_t|/\epsilon)$$

$$\text{REMARK: } \lim_{\epsilon \rightarrow 0} J_\epsilon(\chi, \eta) = j(\chi, \eta) - j(\chi, \nabla_\theta^{-1} \chi)$$

### REGULARIZED VARIATIONAL EQUALITY

FIND A MOTION  $\chi_\epsilon \in \mathcal{V}$ ,  $p \in \mathcal{Q}$  SUCH THAT

$$A(\chi_\epsilon, \eta) - \omega^2 B(\chi_\epsilon, \eta) + C(\chi_\epsilon, \eta) + J_\epsilon(\chi_\epsilon, \eta) +$$

$$I(p, \chi_\epsilon, \eta) = f(\eta) \quad \text{for all } \eta \in \mathcal{V}$$

$$(q, \det(\nabla \chi_\epsilon - 1)) = 0 \quad \text{for all } q \in \mathcal{Q}$$

Figure 8

C1=80 psi C2=20 psi VIS=0 FRIC=.3 DISP= $R_0$  - H = .1 in.

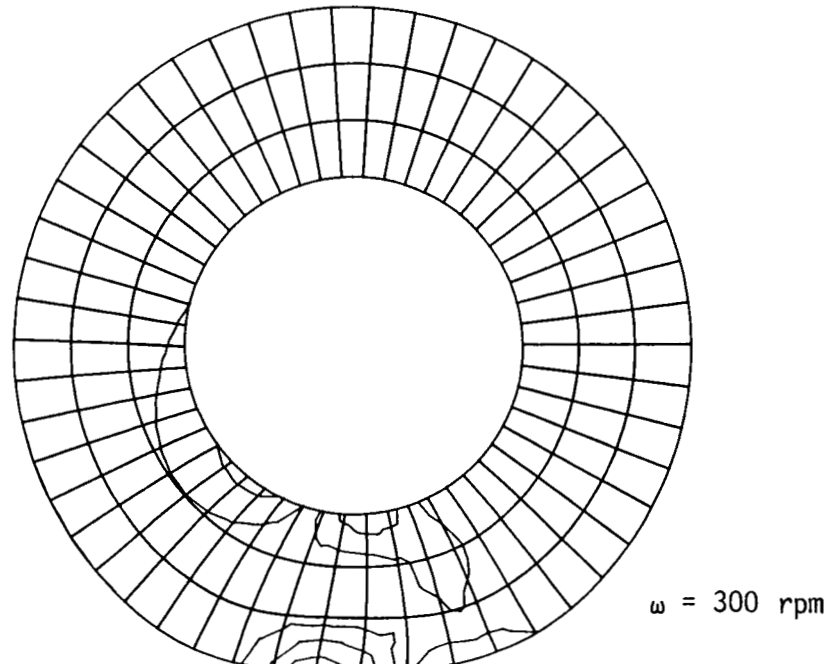


Figure 9

C1=80 psi C2=20 psi VIS=0 FRIC =.3 DISP= $R_0$  - H = .1 in.

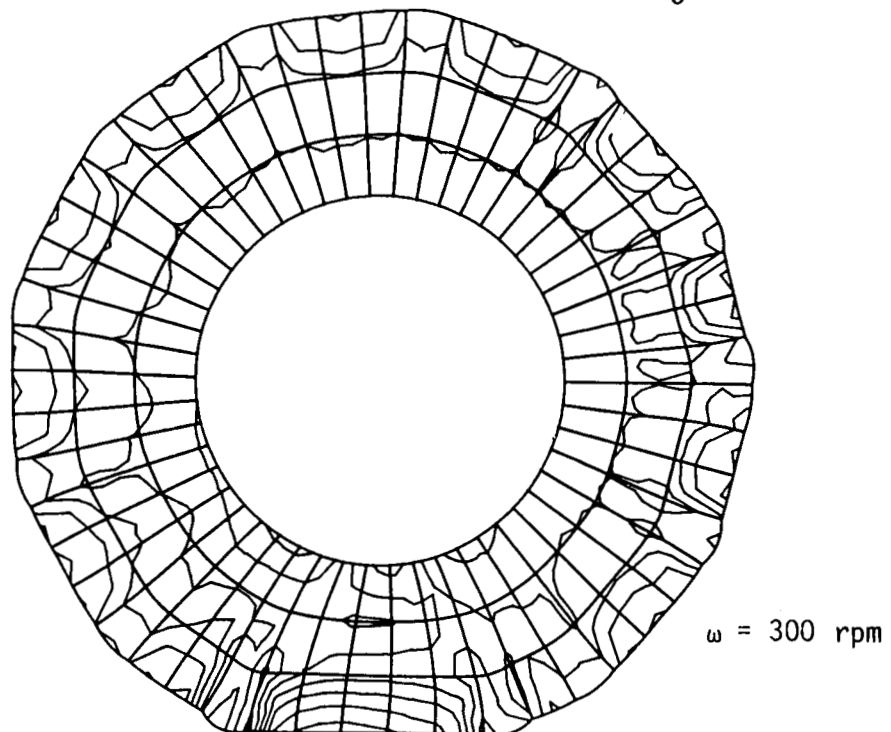


Figure 10



C1=80 psi   C2 = 20 psi   VIS=0   FRIC=.3   DISP= $R_0$  - H = .1 in.

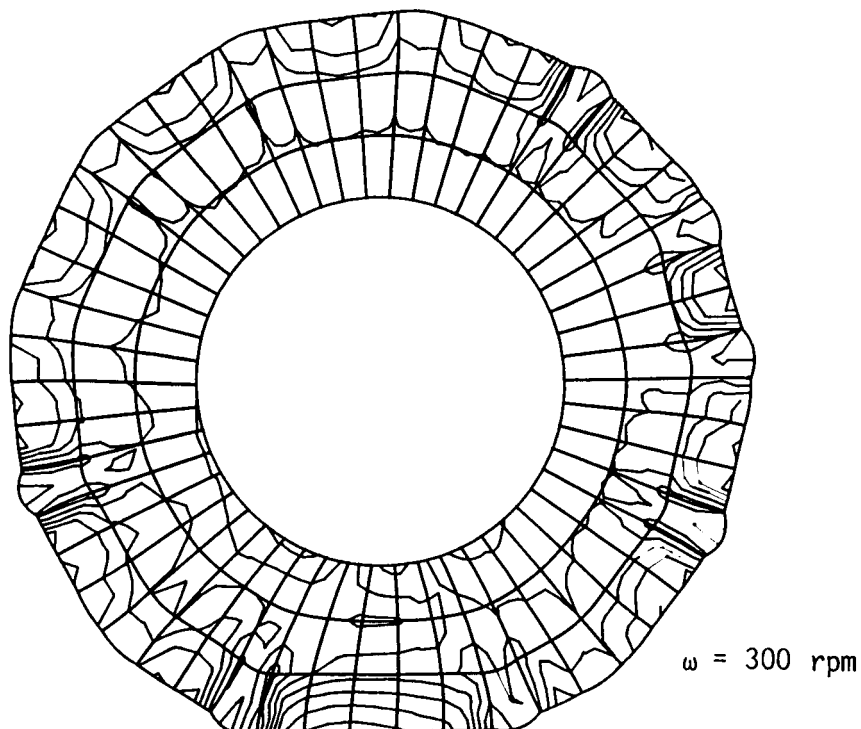


Figure 11

C1=80 psi   C2=20 psi   VIS=0   FRIC=.3   DISP= $R_0$  - H = .1 in.

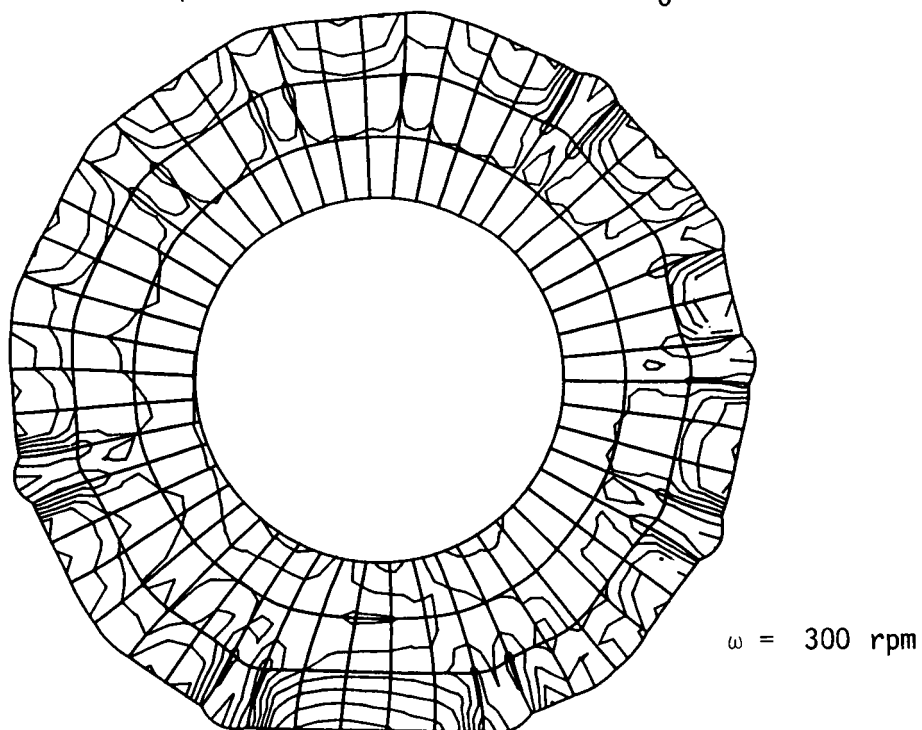


Figure 12

C1=80 psi C2=20 psi VIS=0 FRIC=.3 DISP= $R_0$  - H = .1 in.

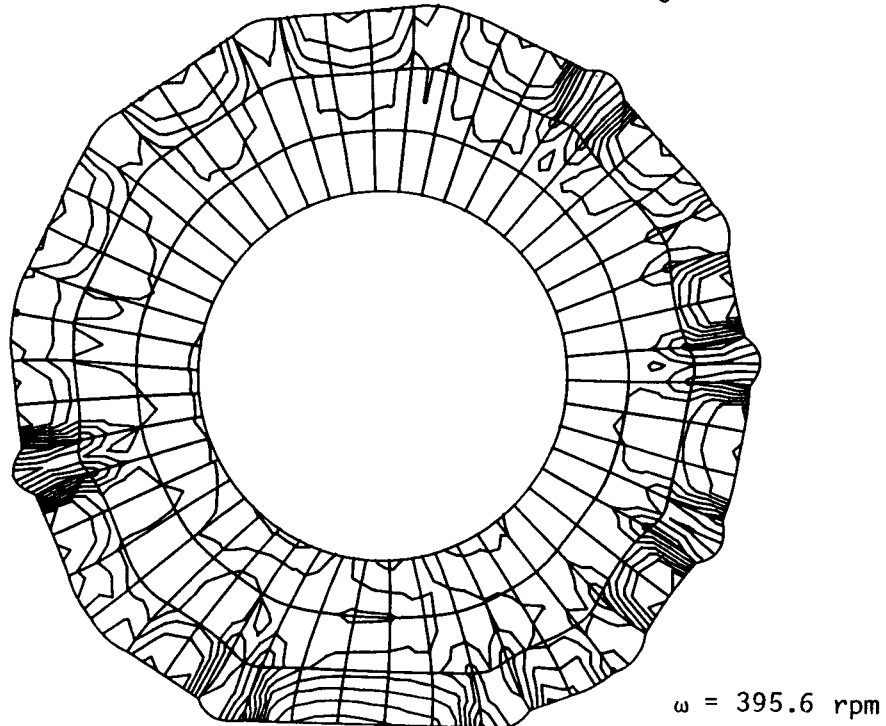
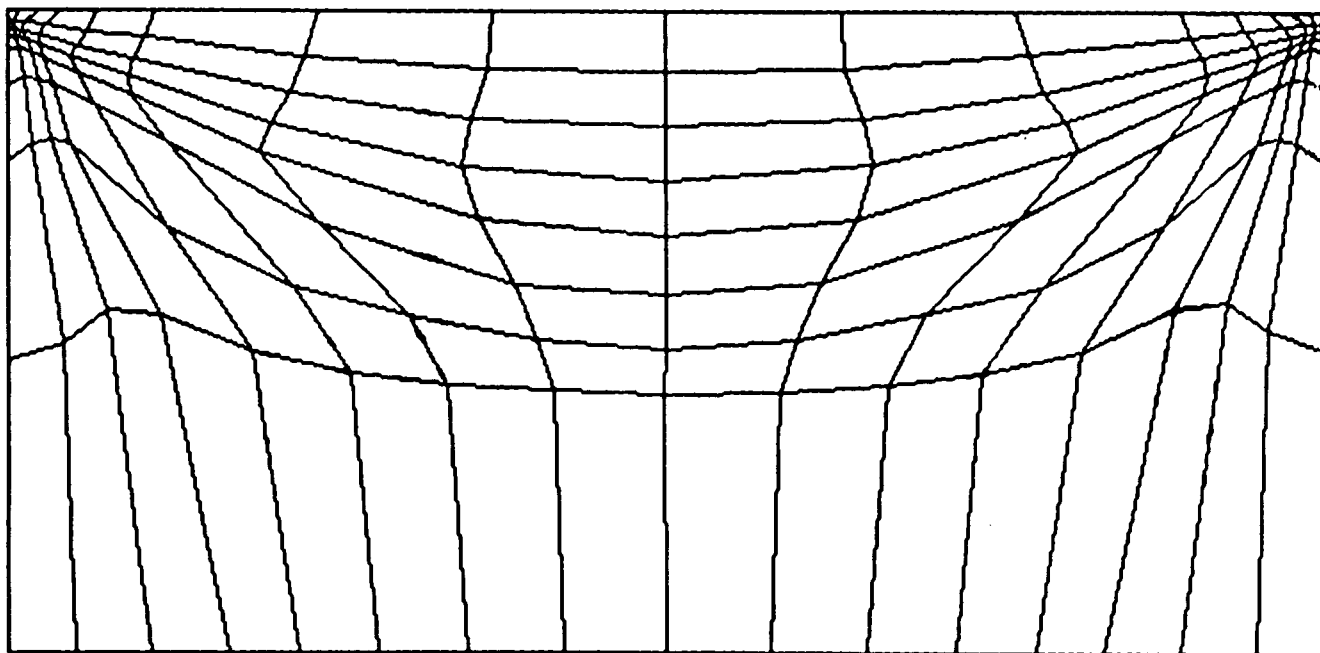


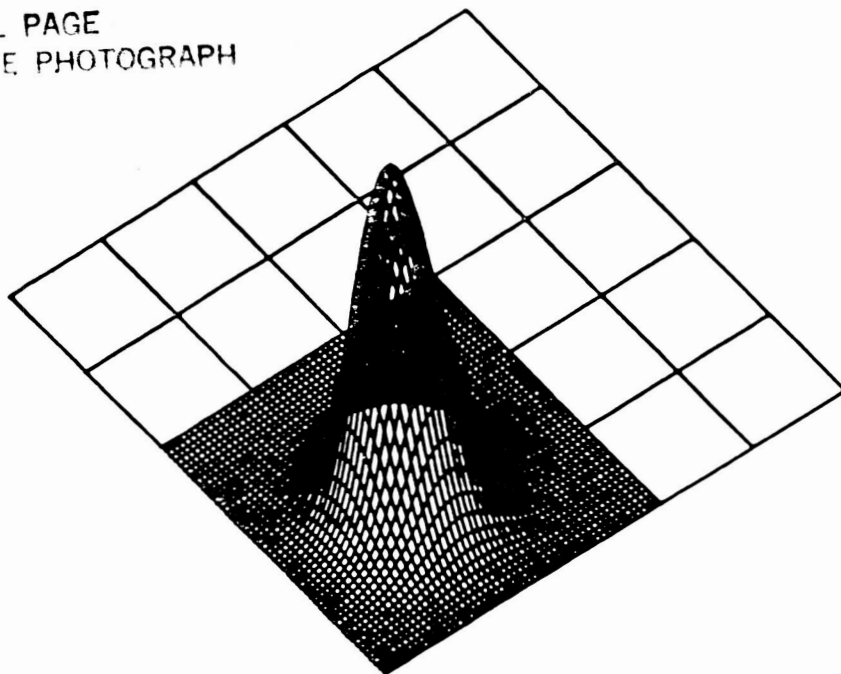
Figure 13



Driven cavity problem. Optimal mesh after 8 FE recalculations.  $\alpha = 4$ ,  $\beta = 4$ ,  $\gamma = 0$

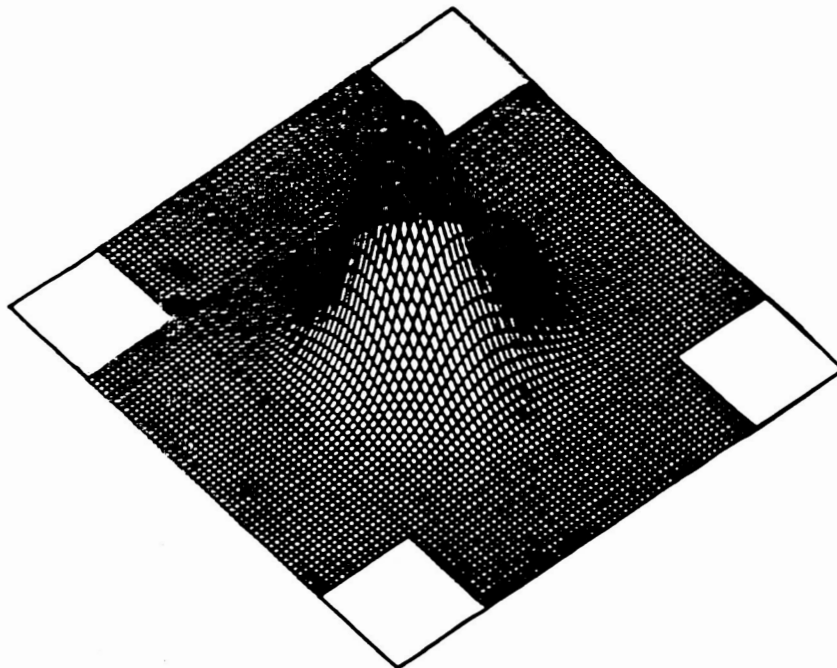
Figure 14

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Heat equation with a dominating convection -  
solution after 1 time step,  $t = 0.02$

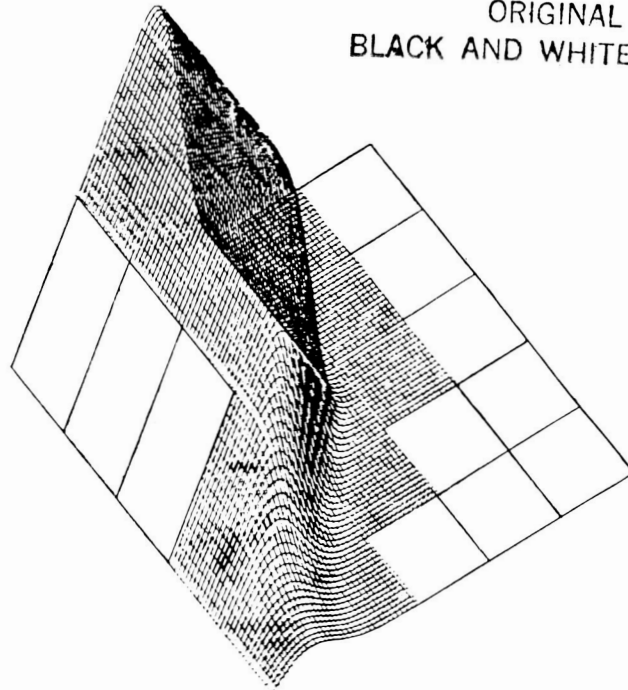
Figure 15



Heat equation with a dominating convection -  
solution after 25 time steps,  $t = 0.5$

Figure 16

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Burger's equations. First component of the solution after 1 time step,  $t = 0.02$

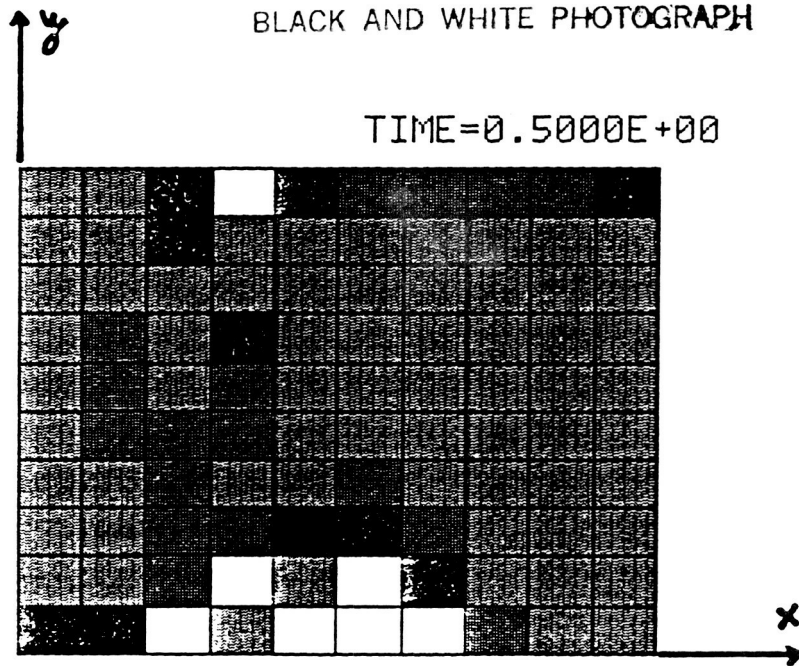
Figure 17



Problem with exponential solution. Deviations of the local effectivity ratios  $\theta_{K,1}^0$  from unity. Parameters:  $h = 2$ ,  $\Delta t = 0.1$ ,  $\delta = 0.25$ ,  $M = 4$

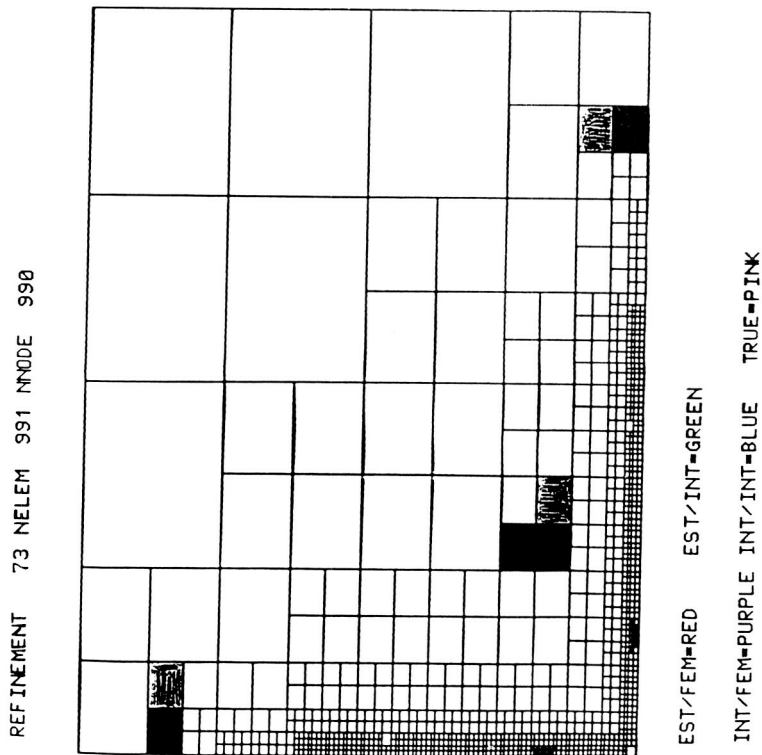
Figure 18

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Problem with exponential solution. Deviations of the local effectivity ratios  $\theta_{K,5}^0$  from unity. Parameters  $h = 2$ ,  $\Delta t = 0.1$ ,  $\delta = 0.5$ ,  $M = 4$

Figure 19



1. Preconditioned Jacobi Conjugate Gradient Scheme
2. 4-Elements/Refinement
3. Rates = Uniform/Const. =  $(C)^{1/2}$

Figure 20